ELSEVIER Journal of Pure and Applied Algebra 126 (1998) 149–168

JOURNAL OF PURE AND APPLIED ALGEBRA

Categorically compact topological groups

Dikran N. Dikranjan[a,*,1], V.V. Uspenskij[b,2]

[a] *Department of Mathematics and Informatics, Udine University, via delle Scienze, 206, 33100 Udine, Italy*

[b] *Department of Mathematics, International Moscow University, Leningradskij prospekt 17, Moscow 125040, Russia*

Communicated by A. Carboni; received 20 December 1995

Abstract

We study the notion of a categorically compact topological group, suggested by the Kuratowski–Mrówka characterization of compact spaces. A topological group G is *categorically compact*, or *C-compact*, if for any topological group H the projection $G\times H \to H$ sends closed subgroups to closed subgroups. We prove, among others, the following theorems: (1) any product of C-compact topological groups is C-compact; (2) separable C-compact groups are totally minimal; (3) C-compact soluble topological groups are compact. ©1998 Elsevier Science B.V.

AMS classification: Primary 22A05; secondary 18A32, 22C05, 54B10, 54D25, 54D30, 54D65

1. Introduction

By the well-known Kuratowski–Mrówka theorem, a topological space G is compact if and only if for every space H the projection $p : G \times H \to H$ is a closed map [27]. This suggests the following

Definition 1.1. A topological group G is *categorically compact* (*C-compact*) if for each topological group H the projection $G\times H \to H$ sends closed subgroups of $G\times H$ to closed subgroups of H.

* Corresponding author. E-mail: dikranja@dimi.uniud.it.

[1] Supported by funds MURST 60%.

[2] Supported by the Alexander von Humboldt Foundation.

PII S0022-4049(96)00139-9

Unless otherwise specified, all topological groups under consideration are assumed to be Hausdorff. Obviously, compact groups are C-compact. We prove that the converse is true for soluble groups (Corollary 3.12). More generally, every C-compact group G contains a closed subgroup H such that the derived subgroup (H,H) is dense in H and the quotient G/H is compact (Theorem 3.10). We conjecture that there exists an infinite discrete C-compact group, but at the moment we have no example of a noncompact C-compact group. A countable discrete group G is C-compact if and only if it is hereditarily totally minimal, i.e. if any quotient of any subgroup of G does not admit a nondiscrete Hausdorff group topology (Theorem 5.5). Thus a discrete simple group with all proper subgroups finite is C-compact iff it is minimal. There exist simple infinite groups with all proper subgroups finite [28], but we do not know if a discrete group with these properties can be minimal.

Compact spaces are characterized among all Tikhonov spaces by each of the following properties: (1) X is compact iff X is absolutely closed, i.e. closed in every Tikhonov space containing it; (2) X is compact iff X is minimal, i.e. X does not admit a strictly coarser Tikhonov (or Hausdorff) topology. Similar properties can be considered for topological groups, but in this case the classes of groups defined by these properties are much wider than the class of compact groups.

Let us say that a topological group G is *absolutely closed* if G is closed in every topological group containing it as a topological subgroup. Recall that G is absolutely closed if and only if it is complete with respect to the upper uniformity (= the upper bound of the left and the right uniformities). Such groups are called Rajkov-complete or sup-complete, and we shall call them simply *complete*, since we do not use the notion of a Weil-complete group.

A topological group is *minimal* if it does not admit a strictly coarser Hausdorff group topology [36]. There are complete minimal groups which are not compact (for example, the unitary group of a Hilbert space has this property [37], the first known example was the infinite symmetric group [8]). All such examples are nonabelian: a fundamental theorem due to Prodanov and Stoyanov [31] implies that abelian minimal complete groups are compact.

We investigate the relations between C-compactness, completeness and minimality for topological groups. We show that every separable C-compact group is minimal (Corollary 3.6). We do not know if separability is essential in this assertion:

Question 1.2. *Is every C-compact group minimal?*

C-compact groups have the following property: if G is C-compact, then every image of G under a continuous homomorphism is complete (Theorem 2.3). This suggests the following

Definition 1.3 (*Dikranjan and Tonolo* [16]). A topological group is *h-complete* if for any continuous onto homomorphism $f : G \to H$ the group H is complete (equivalently, for every continuous homomorphism $f : G \to H$ the subgroup $f(G)$ of H is closed).

Any complete minimal topologically simple group is h-complete (we call a group *topologically simple* if it contains no nontrivial closed normal subgroups). Consequently, h-complete groups need not be compact. For example, for every $n > 1$ the Lie group $SL_n(\mathbb{R})$ is totally minimal ([33] or [14, Theorem 7.4.1]) and thus h-complete. Moreover, every topological group is a subgroup of a complete minimal topologically simple group [40], hence complete groups are precisely closed subgroups of h-complete groups. On the other hand, abelian h-complete groups are compact (Theorem 3.7).

In Section 2 we discuss some basic properties of C-compact groups and h-complete groups. We characterize C-compactness and h-completeness in terms of convergent filters and deduce that the classes of C-compact and h-complete groups are closed under arbitrary products. In Section 3 we prove that separable C-compact groups are minimal and deduce one of our main results: nilpotent h-complete groups and soluble C-compact groups are compact. In Section 4 we introduce another notion of categorical compactness (ν-compactness) which is weaker than h-completeness and study the three-space property for these notions. In Section 5 we study locally compact (in particular, discrete) C-compact groups.

Categorical compactness with respect to an appropriate notion of "closedness" was defined in the case of general categories by Manes [26] (see also [22]). Later, Fay [19] described the categorically compact modules with respect to a hereditary torsion theory. His results were generalized by Giuli and the first named author [12] for arbitrary torsion theories and for a more general concept of "closedness" based on *closure operators* (defined in the case of abstract categories as in [10, 15]). Categorical compactness in such setting was studied in [11] (for topological, Čech and filter convergence spaces), [16] (for topological modules), [32] (for locales) and [4] (for abstract categories). Since then categorical compactness is extensively studied in groups and rings (see [20] and the bibliography there) as well as in abstract categories (see [5] and the bibliography there).

The notation and terminology follow [14, 18, 23]. For groups the multiplicative notation is used, in particular, the neutral element is denoted by 1. If A and B are subgroups of a group G, then (A,B) is the subgroup of G generated by the commutators $x^{-1}y^{-1}xy$, $x \in A$, $y \in B$. The closure of a subset X of a topological space is denoted by $\overline{X}$. Finally, $\mathcal{N}(G)$ denotes the filter of neighbourhoods of unity of a topological group G.

2. Basic properties of C-compact and h-complete groups

The following two propositions are immediate:

Proposition 2.1. *C-compactness is preserved by continuous surjective homomorphisms and by closed subgroups.*

Proposition 2.2. *Finite products of C-compact groups are C-compact.*

Proposition 2.2 will be generalized later to infinite products (Theorem 2.8). We now show that all images of a C-compact group are complete:

Theorem 2.3. *Every C-compact group is h-complete.*

Proof. Let G be a C-compact group. We must show that for every morphism $f : G \to H$ the subgroup $f(G)$ is closed in H. Since the graph of f is a closed subgroup of $G \times H$ which projects onto $f(G)$, we can apply the definition of C-compactness to conclude that $f(G)$ is closed. □

Recall that a space X is compact iff every filter on X has a cluster point or, equivalently, iff every ultrafilter on X converges. A Hausdorff space X is H-closed (= closed in every Hausdorff space containing X as a subspace) if and only if every filter on X with a base of open sets has a cluster point. We give a similar characterization of C-compact and h-complete groups (Theorems 2.7 and 2.11) and deduce that C-compact and h-complete groups are preserved by products (Theorems 2.8 and 2.13).

Definition 2.4. A filter $\mathscr{F}$ on an abstract group G is said to be a *g-filter* if there exists a homomorphism $\varphi : G \to H$ into a topological group H such that for some $h \in H$ the family $\mathscr{B}_{\varphi,h} = \{\varphi^{-1}(U) : U \text{ open in } H,\ h \in U\}$ is a base of $\mathscr{F}$.

Remark 2.5. (a) Note that in the above definition $h \in \overline{\varphi(G)}$.

(b) If $p : G \to G'$ is an onto homomorphism of abstract groups and $\mathscr{F}$ is a g-filter on G', then $p^{-1}(\mathscr{F})$ is a base of a g-filter on G. The fact that g-filters are "transported" also via images is less obvious and will be proved in Proposition 2.6.

Let us say that a family Γ of filters on a set X is *compatible* if Γ is bounded from above in the set of filters on X (equivalently, if the union of Γ has the finite intersection property). It is easy to see that the filter generated by a compatible family of g-filters on a group G is again a g-filter (consider the product of the corresponding topological groups). It follows from the Zorn lemma that every g-filter on a group is contained in a maximal g-filter.

Proposition 2.6. *Let $p : G \to G'$ be an onto homomorphism of abstract groups. Let $\mathscr{F}$ be a g-filter on G. Then the image $p(\mathscr{F})$ of $\mathscr{F}$ under p is a g-filter on G'. Moreover, $p(\mathscr{F})$ is a maximal g-filter on G' whenever $\mathscr{F}$ is a maximal g-filter on G.*

Proof. There exist a topological group H, a homomorphism $\varphi : G \to H$ with $\overline{\varphi(G)} = H$ and an $h \in H$ such that the family $\mathscr{B} = \mathscr{B}_{\varphi,h} = \{\varphi^{-1}(U) : U$ is an open neighbourhood of $h\}$ is a base of $\mathscr{F}$. We identify G' with the quotient group G/N, where N is the kernel of p. Let $H' = H/\overline{\varphi(N)}$, and let $q : H \to H'$ be the quotient map. Let $\psi : G' \to H'$ be the homomorphism defined by $\psi p = q\varphi$. We claim that the image $p(\mathscr{F})$ of $\mathscr{F}$ under p coincides with the g-filter on G' with the base $\mathscr{B}' = \mathscr{B}_{\psi,q(h)}$ (notation as in Definition 2.4).

It suffices to prove that $p(\mathscr{B}) = \mathscr{B}'$. Let $\mathscr{O}$ be the family of all open neighbourhoods of h in H. Then $\mathscr{B}' = \{\psi^{-1}q(U) : U \in \mathscr{O}\}$ and $p(\mathscr{B}) = \{p\varphi^{-1}(U) : U \in \mathscr{O}\}$. We have $\psi^{-1}q(U) = \{gN : \varphi(g) \in U\overline{\varphi(N)}\}$ and $p\varphi^{-1}(U) = \{gN : \varphi(g) \in U\} = \{gN : \varphi(g) \in U\varphi(N)\}$. If U is open, then $U\overline{\varphi(N)} = U\varphi(N)$. Thus $\psi^{-1}q(U) = p\varphi^{-1}(U)$ and hence $p(\mathscr{B}) = \mathscr{B}'$.

Suppose now that $\mathscr{F}$ is a maximal g-filter on G. The first part of our argument shows that $\mathscr{F}' = p(\mathscr{F})$ is a g-filter on G'. Let $\mathscr{J}$ be a g-filter on G' with $\mathscr{F}' \subseteq \mathscr{J}$. We must show that $\mathscr{F}' = \mathscr{J}$. The family $p^{-1}(\mathscr{J})$ is a base of a g-filter on G compatible with $\mathscr{F}$. Hence the family $\{A \cap p^{-1}(B) : A \in \mathscr{F}, B \in \mathscr{J}\}$ is a base of a g-filter $\mathscr{H}$ on G. Since $\mathscr{F}$ is maximal, $\mathscr{H} = \mathscr{F}$. It follows that $\mathscr{J} = p(\mathscr{H}) = p(\mathscr{F}) = \mathscr{F}'$. □

Theorem 2.7. *For a topological group G the following conditions are equivalent:*
(1) *G is C-compact;*
(2) *for every subgroup K of G every g-filter on K has a cluster point in G;*
(3) *for every subgroup K of G every maximal g-filter on K converges in G.*

Proof. (1) ⇒ (2): Assume that G is a C-compact group. Take a subgroup K of G and a g-filter $\mathscr{F}$ on K. There exists a homomorphism $\varphi : K \to H$ into a topological group H such that for some $h \in \overline{\varphi(K)}$ the family $\{\varphi^{-1}(U) : h \in U \subseteq H,\ U \text{ open}\}$ is a base of $\mathscr{F}$. Let $N \subseteq G \times H$ be the graph of φ. Then by the C-compactness of G the projection $p : G \times H \to H$ sends the closure $\overline{N}$ of N onto the closure of $p(N) = \varphi(K)$. Since $h \in \overline{\varphi(K)}$, this yields that for some $g \in G$ one has $(g,h) \in \overline{N}$. Now for each neighbourhood V of g in G and for each neighbourhood U of h in H the subgroup N meets $V \times U$. This means that V meets $\varphi^{-1}(U)$. Hence $g \in \overline{\varphi^{-1}(U)}$. By the choice of $\varphi : G \to H$ and $h \in H$ this yields that g is a cluster point of $\mathscr{F}$.

(2) ⇒ (3): Let $\mathscr{F}$ be a maximal g-filter on a subgroup K of G. Let $h \in G$ be a cluster point of $\mathscr{F}$. We prove that $\mathscr{F}$ converges to h. Let $\mathscr{J}$ be the trace on K of the filter of neighbourhoods of h in G. Then $\mathscr{J}$ is a g-filter on K compatible with $\mathscr{F}$, hence the filter $\mathscr{H}$ generated by $\mathscr{F} \cup \mathscr{J}$ is a g-filter. Since $\mathscr{F}$ is maximal, we have $\mathscr{F} = \mathscr{H} \supseteq \mathscr{J}$. This means that $\mathscr{F}$ converges to h.

(3) ⇒ (2): This is obvious, since every g-filter is contained in a maximal g-filter.

(2) ⇒ (1): We must show that for every topological group H the projection $p : G \times H \to H$ sends each closed subgroup N of the product $G \times H$ to a closed subgroup of H. Let $h \in \overline{p(N)}$. Denote by φ the restriction of p to N. Then $\mathscr{B} = \mathscr{B}_{\varphi,h}$ is a base of a g-filter on N. Let $q : G \times H \to G$ be the projection. Proposition 2.6 implies that $q(\mathscr{B})$ is a base of a g-filter $\mathscr{F}$ on $q(N)$. By the assumption on G, the filter $\mathscr{F}$ has a cluster point $g \in G$. For every neighbourhood U of h in H the set $q\varphi^{-1}(U)$ meets every neighbourhood V of g in G. This means that every neighbourhood $V \times U$ of (g,h) meets N. Since N is closed, we have $(g,h) \in N$ and hence $h \in p(N)$. This proves that $\overline{p(N)} = p(N)$. □

The equivalence of the conditions (1) and (3) of Theorem 2.7 readily implies that C-compactness is preserved by infinite products.

Theorem 2.8. *The product of any family of C-compact groups is C-compact.*

Proof. This follows from Theorem 2.7 and Proposition 2.6. □

Definition 2.9. A filter $\mathscr{F}$ on a topological group is an *og-filter* if it is a g-filter with a base of open sets.

Lemma 2.10. *A filter $\mathscr{F}$ on a topological group G is an og-filter if and only if there exists a continuous homomorphism $\varphi : G \to H$ such that for some $h \in H$ the family $\mathscr{B}_{\varphi,h} = \{\varphi^{-1}(U) : U$ open in H, $h \in U\}$ is a base of $\mathscr{F}$.*

Proof. The "if" part follows immediately from the definitions. To prove the reverse implication, assume that $\mathscr{F}$ is an og-filter on G. There exist a homomorphism (not necessarily continuous) $\varphi : G \to H$ and $h \in H$ such that the family $\mathscr{B}_{\varphi,h}$ is a base for $\mathscr{F}$. It suffices to show that φ is continuous. Let $U \in \mathscr{N}(H)$. Pick $V \in \mathscr{N}(H)$ with $V^{-1}V \subseteq U$. Let $A = \varphi^{-1}(hV)$. Then $A \in \mathscr{F}$. Since $\mathscr{F}$ is an og-filter, the interior of A is not empty. It follows that $A^{-1}A \in \mathscr{N}(G)$. We have $\varphi(A^{-1}A) \subseteq (hV)^{-1}hV = V^{-1}V \subseteq U$. Thus φ is continuous. □

Theorem 2.11. *For a topological group G the following conditions are equivalent:*

(1) *G is h-complete;*
(2) *every og-filter on G has a cluster point;*
(2a) *every og-filter on G is fixed;*
(3) *every maximal og-filter on G converges.*

Proof. (1) ⇒ (2a): This follows from Lemma 2.10.

(2a) ⇒ (2): Trivial.

(2) ⇔ (3): Similar to the proof of Theorem 2.7.

(2) ⇒ (1): Suppose G is not h-complete. Then there exists a continuous homomorphism $\varphi : G \to H$ such that $\varphi(G)$ is not closed. Let $h \in \overline{\varphi(G)} \setminus \varphi(G)$. Then the family $\mathscr{B}_{\varphi,h}$ is a base of an og-filter on G without a cluster point. □

Lemma 2.12. *Let $p : G \to G'$ be a continuous open onto homomorphism of topological groups. The image of every maximal og-filter on G under p is a maximal og-filter on G'.*

Proof. Similar to the proof of Proposition 2.6. □

Just as in the case of C-compactness, we obtain the preservation of h-completeness by products:

Theorem 2.13. *The product of any family of h-complete groups is h-complete.*

Proof. This follows from Theorem 2.11 and Lemma 2.12. □

Note that this theorem (and its proof) can be viewed as a group analogue of the theorem stating that H-closed spaces are preserved by products.

Obviously, h-completeness is preserved by continuous surjective homomorphisms. Example 5.6 shows that it is not preserved by closed normal subgroups. If G is a C-compact group, then all closed subgroups of G are C-compact and hence h-complete. This suggests the following

Question 2.14. *Let G be topological group in which every closed subgroup is h-complete. Must G be C-compact?*

We show that the answer is positive for SIN-groups (Theorem 2.16). A topological group is a SIN-group if the open sets which are invariant under inner automorphisms form a base at the unity. Equivalently, a topological group is SIN iff its left and right uniformities coincide.

Lemma 2.15. *Suppose G is a SIN-group, K is a dense subgroup of G and $\mathscr{F}$ is a g-filter on K. Then the family $\{UA : U \in \mathscr{N}(G), A \in \mathscr{F}\}$ is a base of an og-filter on G.*

Proof. There exist a homomorphism (not necessarily continuous) $\varphi : K \to H$ to a topological group H and $h \in H$ such that $\overline{\varphi(K)} = H$ and the family $\mathscr{B}_{\varphi,h}$ (notation of Definition 2.4) is a base of $\mathscr{F}$. Let $\mathscr{E} = \{U\varphi^{-1}(V) : U \in \mathscr{N}(G), V \in \mathscr{N}(H)\}$. We claim that $\mathscr{E}$ is a neighbourhood base at the unity for some (possibly non-Hausdorff) group topology $\mathscr{T}$ on G.

Let $B = U_1\varphi^{-1}(V_1) \in \mathscr{E}$. We have to check that for every $x \in G$ there exists $C \in \mathscr{E}$ with $C^2 \subseteq B$ and $x^{-1}Cx \subseteq B$. Pick a symmetric $U \in \mathscr{N}(G)$ such that U is invariant under inner automorphisms of G and $U^3 \subseteq U_1$. Pick $y \in K \cap xU$. Let $z = \varphi(y)$. Pick $V \in \mathscr{N}(H)$ with $V^2 \subseteq V_1$ and $z^{-1}Vz \subseteq V_1$. Let $C = U\varphi^{-1}(V)$. Since U is invariant under inner automorphisms, for every $A \subseteq G$ we have $UA = AU$. It follows that $C^2 = U^2\varphi^{-1}(V)^2 \subseteq B$ and $x^{-1}Cx = (x^{-1}y)y^{-1}Cy(y^{-1}x) \subseteq Uy^{-1}CyU = U^3\varphi^{-1}(z^{-1}Vz) \subseteq U_1\varphi^{-1}(V_1) = B$.

Let L be the completion of the Hausdorff group associated with $(G, \mathscr{T})$. The natural homomorphism $p : G \to L$ is continuous, since $\mathscr{T}$ is coarser than the original topology of G. The restriction of $\mathscr{T}$ on K is coarser than the inverse image of the topology of $\varphi(K) \subseteq H$ under φ, hence there exists a continuous homomorphism $q : \varphi(K) \to L$ such that $p|K = q\varphi$. The homomorphism q extends to a homomorphism $H \to L$, which we denote again by q. Let $l = q(h)$. Let $\mathscr{J}$ be the og-filter on G determined by p and l. The family $\mathscr{B}_{p,l}$ is a base of $\mathscr{J}$.

We claim that the filter $\mathscr{C}$ with the base $\{UA : U \in \mathscr{N}(G), A \in \mathscr{F}\}$ equals $\mathscr{J}$. First we check that every $J \in \mathscr{J}$ contains some $C \in \mathscr{C}$. Let $J = p^{-1}(W_1 l)$, where $W_1 \in \mathscr{N}(L)$. Pick $W \in \mathscr{N}(L)$ with $W^2 \subseteq W_1$. Let $U = p^{-1}(W) \in \mathscr{N}(G)$ and $V = q^{-1}(W) \in \mathscr{N}(H)$. Then $J \supseteq p^{-1}(W)p^{-1}(Wl) \supseteq U\varphi^{-1}q^{-1}(Wl) = U\varphi^{-1}(Vh) = C$, where $C \in \mathscr{C}$. Thus $\mathscr{J} \subseteq \mathscr{C}$. Note that $\mathscr{J}$ is a Cauchy filter on $(G, \mathscr{T})$, where the group $(G, \mathscr{T})$

is equipped with the two-sided uniformity. It follows that $\mathscr{C}$ is also a Cauchy filter. We show that $\mathscr{C}$ is a minimal Cauchy filter. According to [3, Chapter 2, Section 3, Proposition 5], it suffices to check that for every $C \in \mathscr{C}$ there exist $D \in \mathscr{C}$ and a neighbourhood W of unity in $(G, \mathscr{T})$ such that $WD \subseteq C$. Let $C = U_1\varphi^{-1}(V_1 h)$, where $U_1 \in \mathscr{N}(G)$ and $V_1 \in \mathscr{N}(H)$. Pick $U \in \mathscr{N}(G)$ and $V \in \mathscr{N}(H)$ so that $U^2 \subseteq U_1$, $V^2 \subseteq V_1$ and U is invariant under inner automorphisms of G. Then $W = U\varphi^{-1}(V)$ and $D = U\varphi^{-1}(Vh)$ have the required properties.

Since $\mathscr{J} \subseteq \mathscr{C}$ and $\mathscr{C}$ is minimal, we have $\mathscr{J} = \mathscr{C}$. Thus $\mathscr{C}$ is an og-filter. □

Theorem 2.16. *A SIN-group G is C-compact if and only if every closed subgroup of G is h-complete.*

Proof. We have noted that every closed subgroup of a C-compact group is h-complete. Let G be a SIN-group such that every closed subgroup of G is h-complete. Let K be a subgroup of G, and let $\mathscr{F}$ be a g-filter on K. According to Theorem 2.7, it suffices to show that $\mathscr{F}$ has a cluster point in G. Replacing G by $\overline{K}$, we may assume that K is dense in G. Consider the family $\mathscr{B} = \{UA : U \in \mathscr{N}(G),\ A \in \mathscr{F}\}$. In virtue of Lemma 2.15, $\mathscr{B}$ is a base of an og-filter on G. Theorem 2.11 implies that $\mathscr{B}$ has a cluster point h in G. It is clear that h is also a cluster point of $\mathscr{F}$. □

We shall see below that h-completeness is not preserved by closed normal subgroups (Example 5.6). We show that it is preserved by closed central subgroups (Proposition 2.18). For this, a lemma is needed:

Lemma 2.17. *Let H be a closed central subgroup of a topological group G. Let $f : H \to H'$ be a surjective continuous homomorphism. Then there exist a topological group G', a closed subgroup H'' of G', a topological isomorphism $i : H' \to H''$ and a continuous surjective homomorphism $F : G \to G'$ such that $i \circ f = F|H$.*

Proof. Let $\mathscr{B}$ be the family of subsets of G of the form $f^{-1}(U)V$, where $U \in \mathscr{N}(H')$ and $V \in \mathscr{N}(G)$. This family is invariant under inner automorphisms of G, and for every $W_1 \in \mathscr{B}$ there exists $W \in \mathscr{B}$ with $W^2 \subset W_1$ (it is here that the assumption that H be central is used). Hence the filter generated by $\mathscr{B}$ is the filter of neighbourhoods of unity for some group topology $\mathscr{T}$ on G (not necessarily Hausdorff). The $\mathscr{T}$-closure of H equals

$$\bigcap\{H \cdot f^{-1}(U)V : U \in \mathscr{N}(H'),\ V \in \mathscr{N}(G)\} = \bigcap\{HV : V \in \mathscr{N}(G)\} = H.$$

Thus H is $\mathscr{T}$-closed in G. Hence H contains the $\mathscr{T}$-closure K of $\{1\}$ in G. Clearly K is a normal subgroup of G. Let G' be the quotient group G/K, and let $F : G \to G'$ be the canonical morphism. Then $K \subseteq H$ implies that $H'' = F(H)$ is a closed subgroup of G'. It is easy to see from the definition of the topology $\mathscr{T}$ that H'' can be identified with the quotient group $(H, \mathscr{T}|H)/K$ and thus there exists a topological isomorphism $i : H' \to H''$ with $i \circ f = F|H$. □

Proposition 2.18. *Every central closed subgroup of a h-complete group is h-complete.*

This result will be strengthened in the next section: the center of a h-complete group is compact.

Proof. Let H be a central closed subgroup of a h-complete group G, and let $f : H \to H'$ be a surjective continuous homomorphism. We must show that H' is complete. In virtue of Lemma 2.17, H' is isomorphic to a closed subgroup of a group G' which is the image of G under a continuous homomorphism. Since G is h-complete, G' is complete, and hence so is H'. □

3. Categorically compact soluble topological groups

In this section we prove our main results: C-compact soluble groups and h-complete nilpotent groups are compact. The question whether h-complete soluble groups are compact remains open.

A topological group G is *precompact* if its completion is compact or, equivalently, if for any $U \in \mathcal{N}(G)$ there is a finite subset $F \subseteq G$ such that $FU = G$. A topological group G is sometimes called *ω-bounded* if for any $U \in \mathcal{N}(G)$ there is a countable subset $A \subseteq G$ such that $AU = G$. In order to avoid confusion we shall call such groups *ω-precompact*. (The term "ω-bounded" has also a different meaning; namely, that closures of countable subsets are compact, for example, the term "ω-bounded" in Theorems 4.2 and 4.4 of the survey [7] should be understood as "ω-precompact".) Obviously, separable groups are ω-precompact. By virtue of a theorem due to Guran (see [1, 21, 39]), a topological group is ω-precompact if and only if it is isomorphic to a subgroup of a product of groups with a countable base.

A topological group G is *totally minimal* if all Hausdorff quotients of G are minimal (see [13]). A group G is totally minimal if and only if every continuous surjective homomorphism $G \to H$ is open. We shall show that ω-precompact C-compact groups are totally minimal (Corollary 3.5). The proof is based upon the Banach Open Mapping Theorem for complete groups with a countable base:

Theorem 3.1 (the Banach open mapping Theorem, Banach [2]). *Every continuous surjective homomorphism between complete groups with a countable base is open.*

See [29] for generalizations of this theorem. For the reader's convenience we sketch the proof.

Let $f : G \to H$ be a continuous surjective homomorphism, where G and H are complete groups with a countable base. For metrizable groups "Rajkov-complete" is equivalent to "completely metrizable", so G and H are Polish spaces. Let $V \in \mathcal{N}(G)$. Pick open $U \in \mathcal{N}(G)$ with $U^{-1}U \subseteq V$. Then $f(U)$ is analytic and hence has the Baire property [25, Section 39, II, Corollary 1], that is can be represented as a symmetric difference of an open set $O \subseteq H$ and a meager set. Since H can be covered by countably many translates of $f(U)$, the set $f(U)$ is not meager itself. It follows that $O \neq \emptyset$. Now it is easy to see that $f(U)f(U)^{-1} \in \mathcal{N}(H)$. Hence $f(V) \in \mathcal{N}(H)$.

Theorem 3.2. *Let G be an ω-precompact h-complete topological group. Then every continuous homomorphism $f : G \to H$ onto a metrizable group H is open.*

Proof. Let $U \in \mathcal{N}(G)$. We must show that $f(U) \in \mathcal{N}(H)$. Since G embeds in the product of separable metrizable groups, we may assume that $U = g^{-1}(V)$ for some continuous homomorphism $g : G \to K$ onto a separable metrizable group K and some $V \in \mathcal{N}(K)$. Let $h = (f,g) : G \to H \times K$, and let $L = h(G)$. Let $p : L \to H$ and $q : L \to K$ be the restrictions to L of the projections $H \times K \to H$ and $H \times K \to K$, respectively. Then $f = ph$ and $g = qh$. Let $W = q^{-1}(V) \in \mathcal{N}(L)$. We have $h(U) = hg^{-1}(V) = hh^{-1}q^{-1}(V) = W$ and $f(U) = ph(U) = p(W)$. Since G is h-complete, the groups L and H, being homomorphic images of G, are complete. They are also separable, being metrizable and ω-precompact. According to Theorem 3.1 the map $p : L \to H$ is open, hence $f(U) = p(W) \in \mathcal{N}(H)$. □

Since every group with a countable network has a weaker metrizable group topology (see [1]), it follows from Theorem 3.2 that every h-complete group G with a countable network is metrizable and minimal. The same is true for any quotient of G, and we get:

Corollary 3.3. *Every h-complete topological group with a countable network is totally minimal and metrizable.* □

Theorem 3.4. *Let G be an ω-precompact topological group such that all closed normal subgroups of G are h-complete. Then G is totally minimal.*

Proof. We must show that for every continuous onto homomorphism $f : G \to H$ and every open $U \in \mathcal{N}(G)$ the set $f(U)$ is open in H. As in the proof of Theorem 3.2, we may assume that $U = g^{-1}(V)$ for some $g : G \to K$ and some $V \in \mathcal{N}(K)$, where K has a countable base. Let L be the kernel of g. Then $U = UL$. Since L is a closed normal subgroup of G and hence h-complete by the assumption, the subgroup $f(L)$ is closed in H and the quotient $H/f(L)$ is Hausdorff. There is a natural homomorphism $\bar{f} : G/L \to H/f(L)$ induced by f. In virtue of Theorem 3.2, the homomorphism $g : G \to K$ is open, hence the quotient G/L is topologically isomorphic to K, and the homomorphism $\bar{f}$ can be considered as a homomorphism of K onto $H/f(L)$. It follows that the group $H/f(L)$ has a countable network. This group is also h-complete, being an image of G, so Corollary 3.3 shows that $H/f(L)$ is metrizable. Applying Theorem 3.2 again, we see that the composition $G \to H \to H/f(L)$ is open, hence the image of $f(U)$ under the quotient map $H \to H/f(L)$ is open in $H/f(L)$. This means that $f(U) = f(UL) = f(U)f(L)$ is open in H. □

Corollary 3.5. *Every C-compact ω-precompact topological group is totally minimal.*

Corollary 3.6. *If G is C-compact, then every closed separable subgroup of G is totally minimal.*

We are now in a position to prove one of our main results:

Theorem 3.7. *Every abelian h-complete topological group is compact.*

Proof. Let G be an abelian h-complete group. Since h-completeness is preserved by closed central subgroups (Proposition 2.18), all closed subgroups of G are h-complete. Consider first the case when G is separable. Then G is ω-precompact, and Theorem 3.4 implies that G is totally minimal. In virtue of the Prodanov–Stoyanov theorem [31], every abelian minimal group is precompact (for totally minimal abelian groups this is due to Prodanov [30]). Since G is also complete, it follows that G is compact. This proves the theorem for separable groups. In the general case, the preceding argument shows that all closed separable subgroups of G are compact. It follows that G is countably compact. Being also complete, it is compact. □

We now generalize Theorem 3.7 to the case of nilpotent groups. Recall that the *upper central series* $\{Z_n(G)\}$ of a group G is defined by: $Z_0(G) = \{1\}$ and $Z_{n+1}(G)/Z_n(G)$ is the center of $G/Z_n(G)$. A group G is *nilpotent* if $Z_n(G) = G$ for some integer n. For a topological group G, the subgroups $Z_n(G)$ are closed.

Theorem 3.8. *For any h-complete topological group G all groups $Z_n(G)$ of the upper central series of G are compact.*

Proof. Since h-completeness is preserved by closed central subgroups (Proposition 2.18), for any n the abelian group $Z_{n+1}(G)/Z_n(G)$, being the centre of an h-complete group $G/Z_n(G)$, is h-complete. Theorem 3.7 implies that the group $Z_{n+1}(G)/Z_n(G)$ is compact. Since the class of compact groups has the three-space property (see Section 4 for the definition), we conclude, by induction, that all the groups $Z_n(G)$ are compact. □

Corollary 3.9. *Every nilpotent h-complete topological group is compact.*

Theorem 3.10. *Let G be a topological group such that all closed subgroups of G are h-complete. Then there is a closed subgroup H of G such that the commutator group (H,H) is dense in H and the quotient G/H is compact.*

Proof. Define the *closed derived series* $\{G^{(\alpha)}\}$ of G as follows: $G^{(0)} = G$, $G^{(\alpha+1)}$ is the closure of the commutator group of $G^{(\alpha)}$ and $G^{(\alpha)} = \bigcap_{\beta<\alpha} G^{(\beta)}$ for limit ordinals α. There is an ordinal γ such that $G^{(\gamma)} = G^{(\gamma+1)}$. We show that $H = G^{(\gamma)}$ is as required.

Since (H,H) is dense in H, it suffices to prove that G/H is compact. Consider first the case when G is ω-precompact. Any quotient $G^{(\alpha)}/G^{(\alpha+1)}$ is abelian and h-complete, hence compact in virtue of Theorem 3.7. We prove by induction on α that the quotient $G/G^{(\alpha)}$ is compact. For non-limit ordinals α this follows from the three-space property (see Section 4) of the class of compact groups. If α is limit, there is a natural injective homomorphism $f_\alpha : G/G^{(\alpha)} \to \prod_{\beta<\alpha} G/G^{(\beta)}$. Since G is h-complete, the range of f_α is closed and hence compact by the assumption of induction. Theorem 3.4 implies that f_α is a homeomorphic embedding of $G/G^{(\alpha)}$. It follows that $G/G^{(\alpha)}$ is compact.

This proves the theorem for the case when G is ω-precompact. In the general case, consider a countable subset D of G/H and let D_1 be a countable subset of G mapped onto D under the canonical homomorphism $G \to G/H$. Then the closed subgroup K of G generated by D is separable. Consider K and its closed derived series $\{K^{(\alpha)}\}$. We have $K^{(\alpha)} \subseteq G^{(\alpha)}$ for any α. In particular, $K^{(\gamma)} \subseteq H$. The group K is ω-precompact and satisfies the assumptions of the theorem, so the first part of the proof shows that the quotient $K/K^{(\gamma)}$ is compact. Hence the image of K in G/H, which is also the image of $K/K^{(\gamma)}$ under the natural map $K/K^{(\gamma)} \to G/H$, is compact. It follows that any countable subset of G/H is contained in a compact set. Thus G/H is countably compact. Being also complete, it is compact. □

The *derived series* $G^{(n)}$ of a group G is defined by: $G^{(0)} = G$ and $G^{(n+1)}$ is the commutator group of $G^{(n)}$. A group G is *soluble* if $G^{(n)} = \{1\}$ for some integer n. Let us say that a topological group G is *topologically hyperabelian* if for every subgroup $H \neq \{1\}$ of G the commutator group (H,H) is not dense in H. A topological group is topologically hyperabelian if and only if its closed derived series, as defined in the proof of Theorem 3.10, arrives at $\{1\}$ at some step, finite or infinite. Every soluble topological group is topologically hyperabelian.

Corollary 3.11. *Every C-compact topologically hyperabelian group is compact.*

Corollary 3.12. *Every C-compact soluble topological group is compact.*

Question 3.13. *Can "C-compact" be replaced by "h-complete" in Corollary 3.12?*

We do not know the answer even for metabelian groups. On the other hand, one can easily see that every h-complete *supersoluble* group (that is a group having a finite normal series with cyclic quotients) is finite, hence compact.

Let us note that for any C-compact group the closed derived series $G^{(\alpha)}$ stabilizes after ω steps: $G^{(\omega)} = G^{(\omega+1)}$. Indeed, we saw in the proof of Theorem 3.10 that the quotient group $K = G/G^{(\omega+1)}$ is compact. The quotient map $G \to K$ sends the series $G^{(\alpha)}$ onto the closed derived series $K^{(\alpha)}$ of K, thus it suffices to prove that $K^{(\omega)} = K^{(\omega+1)}$ for every compact group K. Since every compact group is a projective limit of compact Lie groups, the problem reduces to the case when K is Lie, and in this case the assertion $K^{(\omega)} = K^{(\omega+1)}$ is obvious, since any decreasing sequence of closed subgroups of a compact Lie group is eventually constant.

4. The three-space property and ν-compact groups

A class $\mathscr{K}$ of topological groups has the *three-space property* if the following holds: if H is a closed normal subgroup of G and both H and G/H are in $\mathscr{K}$, then G is in $\mathscr{K}$ [17]. For example, the class of compact groups, the class of metrizable groups and the class of separable groups have this property, while the class of groups with a countable network does not [38]. We show that the class of h-complete groups has the three-space property (Proposition 4.2). This follows from a similar assertion for the class of complete groups:

Lemma 4.1. *The class of complete groups has the three-space property.*

This is a special case of [34, Theorem 12.4]. For the reader's convenience we give a proof.

Proof. If H is a closed normal subgroup of G and $\hat{G}$ is the completion of G, then the quotient group G/H is isomorphic to a dense subgroup of $\hat{G}/\hat{H}$, where $\hat{H}$ is the closure of H in $\hat{G}$ [3, Chapter 3, Section 2, Proposition 21]. Now suppose that H and G/H are complete. Then G/H is both dense and closed in $\hat{G}/\hat{H}$, hence $G/H = \hat{G}/\hat{H}$. Since H is complete, we have $H = \hat{H}$ and thus $G/H = \hat{G}/H$. It follows that $G = \hat{G}$. □

Proposition 4.2. *The class of h-complete groups has the three-space property.*

Proof. Let H be a closed normal subgroup of G such that H and G/H are h-complete. Let $f : G \to G'$ be a continuous surjective homomorphism. We must show that G' is complete. Let $H' = f(H)$. Since H is h-complete, H' is closed in G'. The quotient group G'/H' is also complete, being a homomorphic image of a h-complete group G/H. Lemma 2.7 implies that G' is complete. □

Question 4.3. *Does the class of C-compact groups have the three-space property?*

If N is a compact normal subgroup of G, then G is C-compact iff G/N is C-compact (it suffices to note that the canonical homomorphism $f : G \to G/N$ is a perfect map, so that for every group H the homomorphism $f \times \mathrm{id}_H : G \times H \to G/N \times H$ is a closed map).

Though h-complete groups need not be C-compact, they have the following weaker property naturally related to C-compactness:

Definition 4.4. A topological group G is *ν-compact* if for each topological group H the projection $G \times H \to H$ sends closed normal subgroups of $G \times H$ to closed (normal) subgroups of H.

In the rest of this section we investigate the properties of this weaker notion of compactness. It is easy to see that ν-compact groups are preserved by continuous homomorphisms and by finite products and that every central closed subgroup of a ν-compact group is ν-compact.

Lemma 4.5. *Let H be a h-complete subgroup of a topological group G and N be a closed normal subgroup of G. Then the subgroup NH is closed in G.*

Proof. Let $q : G \to G/N$ be the quotient map. Since H is h-complete, the subgroup $q(H)$ is closed in G/N. Hence $NH = q^{-1}q(H)$ is closed in G. □

Theorem 4.6. *Every h-complete group is v-compact.*

Proof. Let G be a h-complete group. We have to show that for each group H, the projection $p : G\times H \to H$ sends closed normal subgroups of $G\times H$ to closed subgroups of H. Let N be a closed normal subgroup of $G \times H$. Lemma 4.5 implies that NG is closed in $G \times H$ (we consider G as a subgroup of $G \times H$). Since $NG = G \times p(N)$, it follows that $p(N)$ is closed in H. □

We characterize below ν-compactness (Theorem 4.9) and deduce that ν-compact abelian groups are compact.

Lemma 4.7. *For a topological group G the following conditions are equivalent:*

(a) G is v-compact;

(b) for every topological group H the projection $p : G \times H \to H$ sends each closed normal subgroup N of $G \times H$ with $N \cap H = \{1\}$ to a closed subgroup of H.

Proof. The implication (a) ⇒ (b) is trivial. To check (b) ⇒ (a) take a topological group H and a closed normal subgroup N of $G\times H$. Let $N_1 = N \cap H$ and $H' = H/N_1$. The quotient map $q : H \to H'$ is open, and so is the map $\psi = \mathrm{id}\times q : G\times H \to G\times H'$. Since $N = \psi^{-1}\psi(N)$, the normal subgroup $N' = \psi(N)$ is closed in $G \times H'$. On the other hand, $N' \cap H' = \{1\}$, and H' is Hausdorff since N_1 is a closed subgroup of H. Hence our assumption on G yields that the subgroup $p'(N')$ is closed in H', where $p' : G \times H' \to H'$ is the projection. Since $p(N) = q^{-1}p'(N')$, the subgroup $p(N)$ is closed in H. □

We omit an easy verification of the following lemma:

Lemma 4.8. *Let G and K be (abstract) groups, H be a subgroup of G and $f : H \to K$ be a homomorphism. Then the graph of f is a normal subgroup of $G \times K$ if and only if H is a normal subgroup of G, the subgroup $f(H)$ of K is central and (G,H) is contained in the kernel of f.*

Theorem 4.9. *A topological group G is ν-compact if and only if for every closed normal subgroup $H \subset G$ the quotient $H/\overline{(G,H)}$ is compact.*

Proof. We prove the necessity. Suppose that G is ν-compact. Let H be a closed normal subgroup of G. We must prove that the quotient $L = H/\overline{(G,H)}$ is compact. Since L is abelian, in virtue of Theorem 3.7 it suffices to show that L is h-complete. Equivalently, it suffices show that for every continuous homomorphism $f : H \to A$ which is trivial on (G,H) the subgroup $f(H)$ is closed in A. Replacing A by $\overline{f(H)}$ we may assume that $A = \overline{f(H)}$. Then A is abelian since $f(H)$ is so. Let Γ be the graph of f. The subgroup Γ is closed in $H \times A$ and hence also in $G \times A$. According to Lemma 4.8, Γ is a normal subgroup of $G \times A$. Since G is ν-compact, the projection $p : G \times A \to A$ sends Γ to a closed subgroup of A. Thus $f(H) = p(\Gamma)$ is closed in A.

Conversely, suppose that for every closed normal subgroup $H \subset G$ the quotient $H/\overline{(G,H)}$ is compact. We must prove that G is ν-compact. In virtue of Lemma 4.7, it suffices to show that for every topological group K and every closed normal subgroup N of $G \times K$ with $N \cap K = \{1\}$ the subgroup $p(N)$ is closed in K, where $p : G \times K \to K$ is the projection. The group N is the graph of a (not necessarily continuous) homomorphism $f : H \to K$, where H is a normal subgroup of G. Let $L = N \cap G = \operatorname{Ker} f$. Then L is a closed normal subgroup of G, and $L \supseteq (G,H)$ (Lemma 4.8). Let $G' = G/L$, and let $q : G \times K \to G' \times K = (G \times K)/L$ be the quotient map. The subgroup $N' = q(N)$ is closed in $G' \times K$, since $q^{-1}(N') = N$ is closed in $G \times K$. Let $H' = \overline{H}/L$ be the image of $\overline{H}$ in G'. The group H' is compact, since the quotient $\overline{H}/\overline{(G,\overline{H})}$ is compact by the assumption and $(G,\overline{H}) \subset \overline{(G,H)} \subset L$. Let $p' : G' \times K \to K$ be the projection. The restriction of p' to $H' \times K$ is perfect. Since N' is a closed subgroup of $H' \times K$, the subgroup $p'(N')$ of K is closed. It is clear that $p'(N') = p(N)$. Thus $p(N)$ is closed in K. □

Corollary 4.10. *Every topologically simple non-abelian topological group is ν-compact.*

Corollary 4.11. *Every abelian ν-compact group is compact.*

More generally, we have:

Theorem 4.12. *For any ν-compact topological group G all groups $Z_n(G)$ of the upper central series of G are compact.*

The proof is the same as in the case of h-complete groups (Theorem 3.8). It follows that ν-compact nilpotent groups are compact.

Note that for non-abelian groups ν-compactness is substantially weaker than h-completeness (so that, contrary to h-completeness, it does not imply even completeness). Corollary 4.10 provides an easy example of a totally minimal non-complete group which is ν-compact. Let X be an infinite set and let $S(X)$ be the group of all permutations of X. Equip $S(X)$ with the topology of pointwise convergence (considering X

as a discrete space). Let now $A(X)$ be the subgroup of $S(X)$ consisting of all even permutations of X with finite support. Then $A(X)$ is a simple dense totally minimal subgroup ([8, 14, Proposition 7.1.2 and Theorem 7.1.9]), in particular $A(X)$ is not complete. By Corollary 4.10 $A(X)$ is v-compact. Note that $S(X)$ is h-complete, being minimal, complete and topologically simple.

We do not know the answer of the following.

Question 4.13. *Does the class of v-compact groups have the three-space property?*

Nevertheless we can prove:

Proposition 4.14. *If the group G has a normal h-complete subgroup K, then the quotient G/K is v-compact iff G is v-compact.*

Proof. Assume that G/K is v-compact. Let $f : G \to G/K$ be the quotient homomorphism, let H be a topological group and let N be a closed normal subgroup of $G \times H$. In order to prove that for the projection $p : G \times H \to H$ the subgroup $p(N)$ is closed in H we consider the factorization $p = p'\psi$, where $p' : G/K \times H \to H$ is the projection and $\psi = f \times \mathrm{id}_H : G \times H \to G/K \times H$. In virtue of Lemma 4.5 the subgroup NK is closed in $G \times H$. Since $NK \supseteq \ker \psi$, it follows that $\psi(NK)$ is closed in $G/K \times H$. It is also normal, since ψ is surjective. By v-compactness of G/K now $p'(\psi(NK)) = p(N)$ is closed in H. □

5. Locally compact categorically compact groups

In this section we study locally compact, and in particular, discrete, C-compact groups.

Proposition 5.1. *Every connected locally compact C-compact group is compact.*

Proof. According to Corollary 3.12 the group $\mathbb{R}$ (being abelian and non-compact) is not C-compact. Let G be a connected locally compact C-compact group. Then G has no closed subgroup isomorphic to $\mathbb{R}$. By a theorem of Iwasawa (see [24]) G is compact. □

Every totally minimal, locally compact group is obviously h-complete. This remark and Proposition 5.1 imply that h-complete connected Lie groups need not be C-compact. For example, the Lie group $SL_n(\mathbb{R})$ is totally minimal [33] and hence h-complete, but not C-compact. An example with stronger properties will be given below (Example 5.6).

Connectedness is important in Proposition 5.1. In fact, even in the simplest case of discrete groups the question whether C-compact groups are compact (i.e. finite) remains unclear: we do not know if there exists an infinite C-compact discrete group.

Question 5.2. *Is every discrete C-compact group*:
(a) *finite*?
(b) *of finite exponent*?
(c) *finitely generated*?
(d) *countable*?

Note that "yes" to (c) yields that every discrete C-compact group satisfies the maximum condition for subgroups. One can also ask if discrete C-compact groups satisfy the minimum condition for subgroups.

It follows from Corollary 3.12 that every soluble subgroup of a discrete C-compact group G is finite. In particular, G is torsion.

If G is a countable discrete C-compact group, then all subgroups of G are totally minimal (Corollary 3.6). Conversely, if all subgroups of a discrete group G are totally minimal, then G is C-compact (Corollary 5.4).

Note that every totally minimal discrete group is h-complete. Shelah [35] constructed under CH an infinite discrete group which is minimal and simple, hence totally minimal and h-complete. We do not know if an infinite discrete h-complete group is available in ZFC.

Since discrete groups are SIN-groups, Theorem 2.16 implies:

Theorem 5.3. *A discrete group G is C-compact if and only if every subgroup of G is h-complete.*

For a discrete group G the following are equivalent: (1) every g-filter on G is fixed; (2) there is no homomorphism of G onto a proper dense subgroup of a topological group; (3) G is h-complete. These remarks and Theorem 2.7 yield another proof of Theorem 5.3.

Corollary 5.4. *If every subgroup of a discrete group G is totally minimal, then G is C-compact.*

Corollaries 3.6 and 5.4 imply the following criterion for a countable discrete group to be C-compact:

Theorem 5.5. *A countable discrete group G is C-compact if and only if every subgroup of G is totally minimal.*

Example 5.6. There exists an h-complete Lie group G with a closed normal abelian subgroup N which is not h-complete.

Proof. Let G be the semidirect product of $N = \mathbb{R}^2$ and $K = SL_2(\mathbb{R})$ with respect to the natural action $\sigma : N \times K \to N$. Then N is a closed normal abelian subgroup of G. Clearly N is not h-complete. We show that the Lie group G is totally minimal and hence h-complete.

We prove first that G is minimal. To this end we intend to apply Theorem 3.5 of [33] which ensures that the semidirect product G is minimal if four conditions (i) – (iv) are satisfied. In our case (i) and (ii) are trivially satisfied. Let τ be the euclidean topology of N. Condition (iii) consists in asking the action σ to be "minimal" in the sense that every group topology T on N coarser than τ and such that

$$\sigma : (N,T) \times K \to (N,T) \tag{1}$$

is continuous coincides with τ (it is easy to see that this condition is also necessary for the minimality of G). Let D denote the subgroup of diagonal matrices of K with positive entries. Then D leaves invariant $A = \mathbb{R} \times 0$, so induces an action $\sigma' : A \times D \to A$. Let $\pi : D \to \mathbb{R}_+$ be the projection on the first coordinate. Then the action σ' is induced by π and the action σ'' of $\mathbb{R}_+$ on A given by multiplication. Let $A \times_{\sigma''} \mathbb{R}_+$ be the semidirect product obtained in this way. It is isomorphic to the group of upper triangular 2×2 matrices over $\mathbb{R}$ with entries $x, 1$ and $x > 0$ on the diagonal. According to [9] this group is minimal, hence the action σ'' is minimal (in the above sense). Thus the continuity of the restriction of (1) to σ' implies that the restriction of T on A coincides with restriction of τ on A. In particular, for every subset of the form $\Delta \times 0$, where Δ is an open interval of $\mathbb{R}$ containing 0, there is T-neighbourhood U of N with $A \cap U = \Delta \times 0$. On the other hand, an easy compactness argument shows that for each T-neighbourhood U of 0 in N there exists a T-neighbourhood V of 0 in N such that $V' = \sigma(V, SO_n(\mathbb{R})) \subseteq U$. Obviously, V' is $SO_n(\mathbb{R})$-invariant. Since every open ε-ball B_ε with centre 0 in N is also $SO_n(\mathbb{R})$-invariant and $A \cap B_\varepsilon = A \cap U$ for some $SO_n(\mathbb{R})$-invariant T-neighbourhood U of 0 in N, we have proved that B_ε is T-open.

For subsets $C, U \subseteq N$ set $O(C,U) := \{\alpha \in K : (\forall x \in C)\ \alpha(x) - x \in U \cap \alpha(U)\}$. In these terms condition iv) of Theorem 3.5 from [33] says: there exists an open symmetric set $C \subseteq N$ generating N and a neighbourhood U of 0 in N such that if V and U' are neighbourhoods of 0 in N with $O(C,U') \subseteq O(V,U)$, then V is precompact, i.e. a bounded set in N w.r.t. euclidean metric. The verification, similar to that in Example 3.2(b) of [33], will be omitted.

According to [33, Theorem 3.5] the group G is minimal. Moreover, the quotient $G/N \cong SL_n(\mathbb{R})$ is totally minimal according to [33, Theorem 2.4]. Since every non-trivial closed normal subgroup of G contains N, this shows that G is totally minimal. The local compactness of G yields now that G is h-complete. □

Question 5.7. *Is every totally minimal complete group h-complete?*

The answer is positive for metrizable groups and locally compact groups. Note that complete minimal groups may have non-complete quotients. Actually, every topological group is a quotient of a complete minimal group [40].

Remark 5.8. When this paper was finished we learnt that a proof of Theorem 2.8 was obtained independently by Clementino and Tholen [6] in the general setting of categorical compactness.

References

[1] A.V. Arhangel'skiĭ, Classes of topological groups, Russian Math. Surveys 36 (3) (1981) 151–174.
[2] S. Banach, Ueber metrische Gruppen, Studia Math. 3 (1931) 101–113.
[3] N. Bourbaki, General Topology, Part 1 (Addison-Wesley, Reading, MA, 1966).
[4] G. Castellini, Compact objects, surjectivity of epimorphisms and compactifications, Cahiers Topologie Géom. Différentielle Catégoriques, XXXI(1) (1990) 53–65.
[5] M.M. Clementino, Separação e compacidade em categorias, Ph.D. Thesis, Coimbra, 1992.
[6] M.M. Clementino and W. Tholen, Tichonov theorem for categorical compactness, Proc. Amer. Math. Soc., to appear.
[7] W.W. Comfort, K.-H. Hofmann and D. Remus, Topological groups and semigroups, in: M. Hušek and J. van Mill, Ed., Recent Progress in General Topology (Elsevier, Amsterdam, 1992) 315–347.
[8] S. Dierolf and U. Schwanengel, Un example d'un groupe topologique q-minimal mais non precompact, Bull. Sci. Math. 101 (1977) 265–269.
[9] S. Dierolf and U. Schwanengel, Examples of locally compact non-compact minimal topological groups, Pacific J. Math. 82 (1979) 349–355.
[10] D. Dikranjan and E. Giuli, Closure operators I, Topology Appl. 27 (1987) 129–143.
[11] D. Dikranjan and E. Giuli, Compactness, minimality and closedness with respect to a closure operator, in: S. Mac Lane and J. Adamék, Eds., Proc. Internat. Conf. on Categorical Topology, Prague, 1988 (World Scientific, Singapore, 1989) 284–296.
[12] D. Dikranjan and E. Giuli, Factorizations, injectivity and compactness in categories of modules, Comm. Algebra 19 (1991) 45–83.
[13] D. Dikranjan and I. Prodanov, Totally minimal topological groups, Ann. Univ. Sofia Fac. Math. Mec. 69 (1974/75) 5–11.
[14] D. Dikranjan, I. Prodanov and L. Stoyanov, Topological Groups: Characters; Dualities and Minimal Group Topologies, Monographs and Textbooks in Pure and Applied Mathematics, Vol. 130 (Marcel Dekker, New York, 1989).
[15] D. Dikranjan and W. Tholen, Categorical Structure of Closure Operators with Applications to Topology, Algebra and Discrete Mathematics, Mathematics and its Applications, Vol. 346 (Kluwer Academic Publishers, Dordrecht, 1995).
[16] D. Dikranjan and A. Tonolo, On a characterization of linear compactness, Riv. Mat. Pura Appl. 16 (1995) 95–106.
[17] V. Eberhardt, S. Dierolf and U. Schwanengel, On products of two (totally) minimal topological groups and the three-space-problem, Math. Ann. 251 (1980) 123–128.
[18] R. Engelking, General Topology (Heldermann, Berlin, 2nd edn., 1989).
[19] T. Fay, Compact modules, Comm. Algebra 16 (1988) 1209–1219.
[20] T.H. Fay and G.L. Walls, A characterization of categorically compact locally nilpotent groups, Comm. Algebra 22 (1994) 3213–3225.
[21] I. Guran, On topological groups close to being Lindeloff, Soviet Math. Dokl. 23 (1981) 173–175.
[22] H. Herrlich, G. Salicrup and G. Strecker, Factorization, denseness, separation and relatively compact objects, Topology Appl. 27 (1987) 157–169.
[23] E. Hewitt and K.A. Ross, Abstract Harmonic Analysis, Vol. 1 (Springer, Berlin, 2nd edn., 1979).
[24] K. Iwasawa, On some types of topological groups, Ann. Math. 50 (1949) 507–558.
[25] K. Kuratowski, Topology, Vol. 1 (Academic Press, New York, 1966).
[26] E.G. Manes, Compact Hausdorff objects, Gen. Topology Appl. 4 (1974) 341–360.
[27] S. Mrówka, Compactness and product spaces, Colloq. Math. 7 (1959) 19–22.
[28] A. Ol'shanskii, Geometry of defining relations in groups, in: Mathematics and its Applications (Kluwer Academic Publishers, Dordrecht, 1991).
[29] B.J. Pettis, On continuity and openness of homomorphisms in topological groups, Ann. Math. 52 (1950) 293–308.
[30] Iv. Prodanov, Some minimal group topologies are precompact, Math. Ann. 227 (1977) 117–125.
[31] Iv. Prodanov and L. Stoyanov, Every minimal abelian group is precompact, C.R. Acad. Bulgare Sci. 37 (1984) 23–26.
[32] A. Pultr and A. Tozzi, Compactness and Kuratowski theorem for locales, Cahiers Topologie Gèom. Différentielle Catégoriques 33 (1992) 3–14.

[33] D. Remus and L. Stoyanov, Complete minimal and totally minimal groups, Topology Appl. 42 (1991) 57–69.
[34] W. Roelcke and S. Dierolf, Uniform structures on topological groups and their quotients (McGraw-Hill, New York, 1981).
[35] S. Shelah, On a problem of Kurosh, Jonsson groups and applications, in: S.I. Adian, W. W. Boone and G. Higman, Eds., Word Problems II (North-Holland, Amsterdam, 1980) 373–394.
[36] R.M. Stephenson Jr., Minimal topological groups, Math. Ann. 192 (1971) 193–195.
[37] L. Stoyanov, Total minimality of the unitary groups, Math. Z. 187 (1984) 273–283.
[38] V. Uspenskij, Extensions of topological groups with a countable network, Moscow Univ. Math. Bull. 39 (5) (1984) 84–85.
[39] V.V. Uspenskij, Why compact groups are dyadic, in: Z. Frolik, Ed., General Topology and its Relations to Modern Analysis and Algebra VI: Proc. 6th Prague Topological Symp. 1986 (Heldermann, Berlin, 1988) 601–610.
[40] V. Uspenskij, On subgroups of minimal topological groups, in preparation.